

GCSE ONLINE TUTORING × RS REMOTE TUTORING

MATHEMATICS HIGHER

HIGHER

COMPREHENSIVE LESSON NOTEBOOK

UNIT 3: GEOMETRY AND TRIGONOMETRY

Mapped independently to Qualifications Scotland Higher Mathematics C847 76 • 16 detailed chapters
concept teaching • worked methods • reasoning • guided practice • independent practice • worked answers

Student: _____

Class: _____ Target grade: _____ Start date: _____

EXAM-BOARD INDEPENDENCE

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COMPREHENSIVE LESSON NOTEBOOK

How to learn from this notebook

Meaning first. Method second. Reasoning throughout.

1 • UNDERSTAND

Read the definition and big idea. Redraw or annotate the representation, then answer the checkpoint without looking back.

2 • BUILD THE METHOD

Cover each next line of the example. Predict it, reveal it and explain why the transformation or calculation is permitted.

3 • CONNECT AND REASON

Use the misconception analysis and guided task to decide which method fits, not merely how to execute it.

4 • PRACTISE INDEPENDENTLY

Show the decisive setup, accurate progress and a supported conclusion. Do not replace mathematical communication with a calculator display.

5 • DIAGNOSE AND REVISIT

Mark from the worked answer. Record whether an error came from interpretation, structure, arithmetic, accuracy, unit or communication.

COMPREHENSIVE LESSON NOTEBOOK

Geometry and Trigonometry unit map

16 chapters mapped independently to the current course specification.

CH 01	H-G1	Parallel and perpendicular lines
CH 02	H-G2	Gradient and inclination
CH 03	H-G3	Medians, altitudes and perpendicular bisectors
CH 04	H-G4	Equations of circles
CH 05	H-G5	Tangents to circles
CH 06	H-G6	Line-circle intersections
CH 07	H-V1	Vector pathways
CH 08	H-V2	Collinearity
CH 09	H-V3	Internal division points
CH 10	H-V4	Scalar product
CH 11	H-V5	Angles between vectors
CH 12	H-V6	Unit vectors and bases
CH 13	H-T1	Trigonometric equations in degrees and radians
CH 14	H-T2	Addition and double-angle formulae
CH 15	H-T3	Identities and wave form
CH 16	H-G7	Integrated Higher geometry

COMPREHENSIVE LESSON NOTEBOOK

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COMPREHENSIVE LESSON NOTEBOOK

Chapter 1: Parallel and perpendicular lines

Construct equations of related lines and verify perpendicularity.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-G1. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Construct equations of related lines and verify perpendicularity. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find the line through (2,7) parallel to $3x+2y=8$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 1: Build the method

Parallel and perpendicular lines

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find the line through (2,7) parallel to $3x+2y=8$.

WORKED REASONING

1. Keep the same x and y coefficients and substitute the point.

Conclusion: $3x+2y=20$.

WORKED EXAMPLE 2

Find the line through (-1,4) perpendicular to $y=2x-3$.

WORKED REASONING

1. Use perpendicular gradient $-1/2$.

Conclusion: $y=-x/2+7/2$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 1: Connect and reason

Parallel and perpendicular lines

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-G1, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Show $4x-6y=1$ and $3x+2y=9$ are perpendicular. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding parallel and perpendicular lines.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-G1 • Parallel and perpendicular lines

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find the line through (2,7) parallel to $3x+2y=8$.

[4]

2. Find the line through (-1,4) perpendicular to $y=2x-3$.

[4]

3. Show $4x-6y=1$ and $3x+2y=9$ are perpendicular.

[4]

4. Find the intersection of $y=3x-5$ and $y=-x+7$.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-G1 • Parallel and perpendicular lines

1. WORKED ANSWER

Method: Keep the same x and y coefficients and substitute the point. Answer: $3x+2y=20$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Use perpendicular gradient $-1/2$. Answer: $y=-x/2+7/2$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Rearrange each line. Answer: Their gradients are $2/3$ and $-3/2$, whose product is -1 . Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Equate the expressions and solve. Answer: $(3,4)$. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 2: Gradient and inclination

Use $m = \tan(\theta)$ to connect equations and geometric angles.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-G2. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Use $m = \tan(\theta)$ to connect equations and geometric angles. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find the inclination of a line with gradient 1.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 2: Build the method

Gradient and inclination

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find the inclination of a line with gradient 1.

WORKED REASONING

1. $\theta = \tan^{-1}(1)$.

Conclusion: 45 degrees.

WORKED EXAMPLE 2

Find the gradient of a line inclined at 135 degrees.

WORKED REASONING

1. $\tan 135 = -1$.

Conclusion: -1.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 2: Connect and reason

Gradient and inclination

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-G2, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Find the acute angle between lines of gradients 2 and $\frac{1}{3}$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding gradient and inclination.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-G2 • Gradient and inclination

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find the inclination of a line with gradient 1.**[4]**

2. Find the gradient of a line inclined at 135 degrees.**[4]**

3. Find the acute angle between lines of gradients 2 and $\frac{1}{3}$.**[4]**

4. A road rises 7 m for every 120 m horizontally. Find its angle of elevation.**[4]**

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-G2 • Gradient and inclination

1. WORKED ANSWER

Method: $\theta = \tan^{-1}(1)$. Answer: 45 degrees. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: $\tan 135 = -1$. Answer: -1. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Use $\tan \theta = \left| \frac{2 - \frac{1}{3}}{1 + \frac{2}{3}} \right| = 1$. Answer: Approximately 45.0 degrees. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Use $\tan \theta = \frac{7}{120}$. Answer: Approximately 3.34 degrees. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 3: Medians, altitudes and perpendicular bisectors

Apply coordinate methods to triangle centres and geometric loci.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-G3. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Apply coordinate methods to triangle centres and geometric loci. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find the median from $A(1,6)$ in triangle $A(1,6)$, $B(-3,2)$, $C(5,0)$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 3: Build the method

Medians, altitudes and perpendicular bisectors

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find the median from A(1,6) in triangle A(1,6), B(-3,2), C(5,0).

WORKED REASONING

1. The midpoint of BC is (1,1).
2. The line through A(1,6) and the midpoint is vertical.

Conclusion: $x=1$.

WORKED EXAMPLE 2

Find the perpendicular bisector of P(2,1), Q(8,5).

WORKED REASONING

1. Midpoint is (5,3); PQ gradient is $\frac{2}{3}$, so perpendicular gradient is $-\frac{3}{2}$.
2. Use $y-3=-\frac{3}{2}(x-5)$.

Conclusion: $y=-\frac{3x}{2}+\frac{21}{2}$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 3: Connect and reason

Medians, altitudes and perpendicular bisectors

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-G3, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Find the altitude from $A(0,4)$ to BC where $B(-2,0)$, $C(6,2)$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding medians, altitudes and perpendicular bisectors.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-G3 • Medians, altitudes and perpendicular bisectors

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find the median from $A(1,6)$ in triangle $A(1,6)$, $B(-3,2)$, $C(5,0)$.

[4]

2. Find the perpendicular bisector of $P(2,1)$, $Q(8,5)$.

[4]

3. Find the altitude from $A(0,4)$ to BC where $B(-2,0)$, $C(6,2)$.

[4]

4. Explain why the perpendicular bisectors of a triangle meet at a point equidistant from all vertices.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-G3 • Medians, altitudes and perpendicular bisectors

1. WORKED ANSWER

Method: The midpoint of BC is (1,1). The line through A(1,6) and the midpoint is vertical. Answer: $x=1$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Midpoint is (5,3); PQ gradient is $\frac{2}{3}$, so perpendicular gradient is $-\frac{3}{2}$. Use $y-3=-\frac{3}{2}(x-5)$. Answer: $y=-\frac{3x}{2}+21/2$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: BC gradient is $\frac{1}{4}$; perpendicular gradient is -4 . Answer: $y=-4x+4$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Use the locus definition. Answer: Every point on a perpendicular bisector is equidistant from that side's endpoints; the intersection satisfies this for two sides and hence all three vertices. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 4: Equations of circles

Build and transform circle equations.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-G4. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Build and transform circle equations. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “State centre and radius of $(x+4)^2+(y-3)^2=49$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 4: Build the method

Equations of circles

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

State centre and radius of $(x+4)^2 + (y-3)^2 = 49$.

WORKED REASONING

1. Compare with standard form.

Conclusion: Centre $(-4,3)$, radius 7.

WORKED EXAMPLE 2

Find the circle with diameter endpoints $(-2,1)$ and $(6,7)$.

WORKED REASONING

1. Centre is midpoint $(2,4)$; diameter length is 10, so radius 5.

Conclusion: $(x-2)^2 + (y-4)^2 = 25$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 4: Connect and reason

Equations of circles

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-G4, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Write $x^2 + y^2 + 8x - 6y - 11 = 0$ in standard form. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding equations of circles.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-G4 • Equations of circles

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. State centre and radius of $(x+4)^2+(y-3)^2=49$.

[4]

2. Find the circle with diameter endpoints $(-2,1)$ and $(6,7)$.

[4]

3. Write $x^2+y^2+8x-6y-11=0$ in standard form.

[4]

4. Find k if $(5,-1)$ lies on $x^2+y^2-2x+4y+k=0$.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-G4 • Equations of circles

1. WORKED ANSWER

Method: Compare with standard form. Answer: Centre $(-4,3)$, radius 7. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Centre is midpoint $(2,4)$; diameter length is 10, so radius 5. Answer: $(x-2)^2+(y-4)^2=25$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Complete both squares. Answer: $(x+4)^2+(y-3)^2=36$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Substitute the point: $25+1-10-4+k=0$. Answer: $k=-12$. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 5: Tangents to circles

Use radius-tangent perpendicularity and distance conditions.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-G5. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Use radius-tangent perpendicularity and distance conditions. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find the tangent to $x^2 + y^2 = 25$ at $(3,4)$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 5: Build the method

Tangents to circles

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find the tangent to $x^2 + y^2 = 25$ at (3,4).

WORKED REASONING

1. The radius is normal to the tangent.

Conclusion: $3x + 4y = 25$.

WORKED EXAMPLE 2

Find tangents to $x^2 + y^2 = 9$ parallel to $y = 2x$.

WORKED REASONING

1. Distance from the origin to $2x - y + c = 0$ must equal 3.

Conclusion: $y = 2x + 3\sqrt{5}$ and $y = 2x - 3\sqrt{5}$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 5: Connect and reason

Tangents to circles

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-G5, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Show $y = x + \sqrt{2}$ is tangent to $x^2 + y^2 = 1$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding tangents to circles.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-G5 • Tangents to circles

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find the tangent to $x^2+y^2=25$ at (3,4).

[4]

2. Find tangents to $x^2+y^2=9$ parallel to $y=2x$.

[4]

3. Show $y=x+\sqrt{2}$ is tangent to $x^2+y^2=1$.

[4]

4. A tangent from P is 12 cm and $OP=13$ cm, where O is the centre. Find the radius.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-G5 • Tangents to circles

1. WORKED ANSWER

Method: The radius is normal to the tangent. Answer: $3x+4y=25$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Distance from the origin to $2x-y+c=0$ must equal 3. Answer: $y=2x+3\sqrt{5}$ and $y=2x-3\sqrt{5}$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: One repeated intersection proves tangency. Answer: Substitution gives $2x^2+2\sqrt{2}x+1=0$ with discriminant 0. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Radius to tangent point is perpendicular; use Pythagoras. Answer: 5 cm. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 6: Line-circle intersections

Solve and interpret intersections of lines and circles.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-G6. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Solve and interpret intersections of lines and circles. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find where $y=3$ meets $x^2+y^2=25$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 6: Build the method

Line-circle intersections

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find where $y=3$ meets $x^2+y^2=25$.

WORKED REASONING

1. Substitute $y=3$.

Conclusion: $(-4,3)$ and $(4,3)$.

WORKED EXAMPLE 2

Find intersections of $y=x+1$ and $x^2+y^2=13$.

WORKED REASONING

1. Substitute and solve $x^2+(x+1)^2=13$.

Conclusion: $(-3,-2)$ and $(2,3)$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 6: Connect and reason

Line-circle intersections

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-G6, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Determine whether $y=2x+5$ meets $x^2+y^2=4$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding line-circle intersections.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-G6 • Line-circle intersections

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find where $y=3$ meets $x^2+y^2=25$.

[4]

2. Find intersections of $y=x+1$ and $x^2+y^2=13$.

[4]

3. Determine whether $y=2x+5$ meets $x^2+y^2=4$.

[4]

4. Find k so $y=k$ is tangent to $(x-2)^2+(y+1)^2=16$.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-G6 • Line-circle intersections

1. WORKED ANSWER

Method: Substitute $y=3$. Answer: $(-4,3)$ and $(4,3)$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Substitute and solve $x^2+(x+1)^2=13$. Answer: $(-3,-2)$ and $(2,3)$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Substitution gives a quadratic with negative discriminant. Answer: No real intersection. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Horizontal tangents are one radius above and below the centre. Answer: $k=3$ or $k=-5$. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 7: Vector pathways

Determine resultants of connected vector pathways in three dimensions.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-V1. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Determine resultants of connected vector pathways in three dimensions. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “If $AB=(2,-1,4)$ and $BC=(5,3,-2)$, find AC .”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 7: Build the method

Vector pathways

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

If $AB = (2, -1, 4)$ and $BC = (5, 3, -2)$, find AC .

WORKED REASONING

1. Add the pathway vectors.

Conclusion: $(7, 2, 2)$.

WORKED EXAMPLE 2

If $OA = a$, $OB = b$ and C divides AB with $AC:CB = 2:1$, write OC .

WORKED REASONING

1. Move two thirds of the way from A to B .

Conclusion: $(a + 2b)/3$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 7: Connect and reason

Vector pathways

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-V1, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

A closed path uses vectors p , q , r and s . Express s . Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding vector pathways.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-V1 • Vector pathways

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. If $\mathbf{AB}=(2,-1,4)$ and $\mathbf{BC}=(5,3,-2)$, find $\mathbf{AC}$.

[4]

2. If $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=\mathbf{b}$ and C divides AB with $\mathbf{AC:CB}=2:1$, write $\mathbf{OC}$.

[4]

3. A closed path uses vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$. Express $\mathbf{s}$.

[4]

4. Given $\mathbf{PQ}=(3,4,-1)$ and $\mathbf{PR}=(-2,5,6)$, find $\mathbf{QR}$.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-V1 • Vector pathways

1. WORKED ANSWER

Method: Add the pathway vectors. Answer: $(7,2,2)$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Move two thirds of the way from A to B. Answer: $(a+2b)/3$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: The vector sum around a closed path is zero. Answer: $s=-(p+q+r)$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: $QR=PR-PQ$. Answer: $(-5,1,7)$. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 8: Collinearity

Prove points are collinear and identify directional relationships.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-V2. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Prove points are collinear and identify directional relationships. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Show $A(1,2,3)$, $B(3,6,7)$, $C(4,8,9)$ are collinear.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 8: Build the method

Collinearity

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Show $A(1,2,3)$, $B(3,6,7)$, $C(4,8,9)$ are collinear.

WORKED REASONING

1. Parallel displacement vectors from a common point prove collinearity.

Conclusion: $\mathbf{AB}=(2,4,4)$ and $\mathbf{AC}=(3,6,6)=1.5\mathbf{AB}$.

WORKED EXAMPLE 2

Determine whether $\mathbf{u}=(2,-3,5)$ and $\mathbf{v}=(-4,6,-10)$ are parallel.

WORKED REASONING

1. Each component has the same scalar multiplier.

Conclusion: Yes; $\mathbf{v}=-2\mathbf{u}$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 8: Connect and reason

Collinearity

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-V2, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Find k so $(1, k, 7)$ is parallel to $(2, 6, 14)$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding collinearity.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-V2 • Collinearity

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Show $A(1,2,3)$, $B(3,6,7)$, $C(4,8,9)$ are collinear.

[4]

2. Determine whether $u=(2,-3,5)$ and $v=(-4,6,-10)$ are parallel.

[4]

3. Find k so $(1,k,7)$ is parallel to $(2,6,14)$.

[4]

4. Explain why three points with $AB=\lambda AC$ are collinear.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-V2 • Collinearity

1. WORKED ANSWER

Method: Parallel displacement vectors from a common point prove collinearity. Answer: $AB=(2,4,4)$ and $AC=(3,6,6)=1.5AB$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Each component has the same scalar multiplier. Answer: Yes; $v=-2u$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: The scale factor is $1/2$. Answer: $k=3$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: State the scalar-multiple condition. Answer: Both displacement vectors have the same or opposite direction, so B and C lie on the same straight line through A. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 9: Internal division points

Find coordinates of points dividing a segment in a given ratio.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-V3. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Find coordinates of points dividing a segment in a given ratio. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find the midpoint of A(-2,5,1), B(6,-1,9).”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 9: Build the method

Internal division points

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find the midpoint of $A(-2,5,1)$, $B(6,-1,9)$.

WORKED REASONING

1. Average components.

Conclusion: $(2,2,5)$.

WORKED EXAMPLE 2

P divides AB in ratio $AP:PB=1:3$, $A=(2,4)$, $B=(10,0)$. Find P.

WORKED REASONING

1. P is one quarter of the way from A to B.

Conclusion: $(4,3)$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 9: Connect and reason

Internal division points

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-V3, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Q divides AB in ratio $AQ:QB=2:1$, $A=(1,-2,4)$, $B=(7,4,10)$. Find Q. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding internal division points.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-V3 • Internal division points

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find the midpoint of $A(-2,5,1)$, $B(6,-1,9)$.

[4]

2. P divides AB in ratio $AP:PB=1:3$, $A=(2,4)$, $B=(10,0)$. Find P.

[4]

3. Q divides AB in ratio $AQ:QB=2:1$, $A=(1,-2,4)$, $B=(7,4,10)$. Find Q.

[4]

4. Point $P=(3,2)$ divides $A=(1,1)$, $B=(9,5)$. Find $AP:PB$.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-V3 • Internal division points

1. WORKED ANSWER

Method: Average components. Answer: (2,2,5). Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: P is one quarter of the way from A to B. Answer: (4,3). Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Use $(A+2B)/3$. Answer: (5,2,8). Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: $P-A=(2,1)$ and $B-P=(6,3)=3(2,1)$. Answer: 1:3. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 10: Scalar product

Evaluate dot products and use orthogonality.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-V4. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Evaluate dot products and use orthogonality. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find $(2,-1,3)$ dot $(4,5,-2)$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 10: Build the method

Scalar product

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find $(2, -1, 3)$ dot $(4, 5, -2)$.

WORKED REASONING

1. Calculate $8 - 5 - 6$.

Conclusion: -3.

WORKED EXAMPLE 2

Show $(1, 2, -1)$ is perpendicular to $(3, -1, 1)$.

WORKED REASONING

1. A zero scalar product means perpendicular.

Conclusion: Their dot product is $3 - 2 - 1 = 0$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 10: Connect and reason

Scalar product

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-V4, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Find k if $(k, 2, 1)$ is perpendicular to $(3, -1, 4)$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding scalar product.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-V4 • Scalar product

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find $(2, -1, 3)$ dot $(4, 5, -2)$.

[4]

2. Show $(1, 2, -1)$ is perpendicular to $(3, -1, 1)$.

[4]

3. Find k if $(k, 2, 1)$ is perpendicular to $(3, -1, 4)$.

[4]

4. Find a dot a for $a = (5, -2, 4)$.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-V4 • Scalar product

1. WORKED ANSWER

Method: Calculate $8-5-6$. Answer: -3 . Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: A zero scalar product means perpendicular. Answer: Their dot product is $3-2-1=0$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Set $3k-2+4=0$. Answer: $k=-2/3$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: This equals $|a|^2$. Answer: 45. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 11: Angles between vectors

Use the scalar-product formula to determine angles.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-V5. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Use the scalar-product formula to determine angles. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find the angle between $(1,0)$ and $(1,1)$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 11: Build the method

Angles between vectors

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find the angle between $(1,0)$ and $(1,1)$.

WORKED REASONING

1. $\cos \theta = 1/(1 \times \sqrt{2})$.

Conclusion: 45 degrees.

WORKED EXAMPLE 2

Find the angle between $a=(2,-1,2)$ and $b=(1,2,2)$.

WORKED REASONING

1. $a \cdot b = 4$; both magnitudes are 3, so $\cos \theta = 4/9$.

2. Therefore $\theta = \cos^{-1}(4/9)$, approximately 63.6 degrees.

Conclusion: Approximately 63.6 degrees.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 11: Connect and reason

Angles between vectors

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-V5, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Determine whether (3,4) and (-6,-8) point in the same or opposite direction. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding angles between vectors.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-V5 • Angles between vectors

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find the angle between $(1,0)$ and $(1,1)$.

[4]

2. Find the angle between $a=(2,-1,2)$ and $b=(1,2,2)$.

[4]

3. Determine whether $(3,4)$ and $(-6,-8)$ point in the same or opposite direction.

[4]

4. A rhombus has adjacent side vectors a and b with equal magnitude. Explain a condition for it to be a square.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-V5 • Angles between vectors

1. WORKED ANSWER

Method: $\cos \theta = 1/(1 \times \sqrt{2})$. Answer: 45 degrees. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: $a \cdot b = 4$; both magnitudes are 3, so $\cos \theta = 4/9$. Therefore $\theta = \cos^{-1}(4/9)$, approximately 63.6 degrees. Answer: Approximately 63.6 degrees. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: The second vector is -2 times the first; angle 180 degrees. Answer: Opposite direction. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Adjacent sides must also be perpendicular. Answer: $a \cdot b = 0$. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 12: Unit vectors and bases

Find unit vectors and express vectors using i , j and k .

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-V6. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Find unit vectors and express vectors using i , j and k . The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find a unit vector in the direction $(3,4)$.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 12: Build the method

Unit vectors and bases

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find a unit vector in the direction $(3,4)$.

WORKED REASONING

1. Divide by magnitude 5.

Conclusion: $(3/5, 4/5)$.

WORKED EXAMPLE 2

Write the vector from $A(1, -2, 4)$ to $B(5, 3, 1)$ using i, j, k .

WORKED REASONING

1. Subtract coordinates $B - A$.

Conclusion: $4i + 5j - 3k$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 12: Connect and reason

Unit vectors and bases

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-V6, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Find a vector of magnitude 10 in the direction $(2, -1, 2)$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding unit vectors and bases.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-V6 • Unit vectors and bases

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find a unit vector in the direction $(3,4)$.

[4]

2. Write the vector from $A(1,-2,4)$ to $B(5,3,1)$ using i,j,k .

[4]

3. Find a vector of magnitude 10 in the direction $(2,-1,2)$.

[4]

4. If $\mathbf{a}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{b}=\mathbf{i}+4\mathbf{j}-\mathbf{k}$, find $2\mathbf{a}-\mathbf{b}$.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-V6 • Unit vectors and bases

1. WORKED ANSWER

Method: Divide by magnitude 5. Answer: $(3/5, 4/5)$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Subtract coordinates B-A. Answer: $4i+5j-3k$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: The direction magnitude is 3; multiply the unit vector by 10. Answer: $(20/3, -10/3, 20/3)$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Combine coefficients componentwise. Answer: $3i-6j+7k$. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 13: Trigonometric equations in degrees and radians

Solve trigonometric equations over stated intervals and report every valid solution.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-T1. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Solve trigonometric equations over stated intervals and report every valid solution. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Solve $\sin x = 1/2$ for $0 \leq x < 360$ degrees.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 13: Build the method

Trigonometric equations in degrees and radians

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Solve $\sin x = 1/2$ for $0 \leq x < 360$ degrees.

WORKED REASONING

1. Use the reference angle 30 degrees.
2. Sine is positive in quadrants I and II.

Conclusion: $x = 30$ degrees or $x = 150$ degrees.

WORKED EXAMPLE 2

Solve $2\cos x + 1 = 0$ for $0 \leq x \leq 2\pi$.

WORKED REASONING

1. First obtain $\cos x = -1/2$.
2. Cosine is negative in quadrants II and III.

Conclusion: $x = 2\pi/3$ or $x = 4\pi/3$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 13: Connect and reason

Trigonometric equations in degrees and radians

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-T1, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Solve $\tan(2x)=1$ for $0 \leq x < 180$ degrees. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding trigonometric equations in degrees and radians.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-T1 • Trigonometric equations in degrees and radians

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Solve $\sin x = 1/2$ for $0 \leq x < 360$ degrees.

[4]

2. Solve $2\cos x + 1 = 0$ for $0 \leq x \leq 2\pi$.

[4]

3. Solve $\tan(2x) = 1$ for $0 \leq x < 180$ degrees.

[4]

4. Why must calculator angle mode be checked before solving a trigonometric equation?

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-T1 • Trigonometric equations in degrees and radians

1. WORKED ANSWER

Method: Use the reference angle 30 degrees. Sine is positive in quadrants I and II. Answer: $x=30$ degrees or $x=150$ degrees. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: First obtain $\cos x = -1/2$. Cosine is negative in quadrants II and III. Answer: $x=2\pi/3$ or $x=4\pi/3$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: The corresponding interval for $2x$ is $0 \leq 2x < 360$ degrees. Solve $2x=45$ degrees or 225 degrees, then divide by 2. Answer: $x=22.5$ degrees or $x=112.5$ degrees. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Confirm the mode before evaluating inverse trigonometric functions. Answer: Degree and radian modes interpret the same input differently, so the mode must match the interval and units in the question. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 14: Addition and double-angle formulae

Apply angle-sum and double-angle formulae exactly.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-T2. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Apply angle-sum and double-angle formulae exactly. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Find $\sin 75$ degrees exactly.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 14: Build the method

Addition and double-angle formulae

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Find $\sin 75^\circ$ exactly.

WORKED REASONING

1. Use $\sin(45+30)$.

Conclusion: $(\sqrt{6}+\sqrt{2})/4$.

WORKED EXAMPLE 2

Find $\cos 15^\circ$ exactly.

WORKED REASONING

1. Use $\cos(45-30)$.

Conclusion: $(\sqrt{6}+\sqrt{2})/4$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 14: Connect and reason

Addition and double-angle formulae

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-T2, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Given $\sin x = 3/5$ and $\cos x = 4/5$, find $\sin 2x$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding addition and double-angle formulae.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-T2 • Addition and double-angle formulae

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Find $\sin 75^\circ$ exactly.

[4]

2. Find $\cos 15^\circ$ exactly.

[4]

3. Given $\sin x = \frac{3}{5}$ and $\cos x = \frac{4}{5}$, find $\sin 2x$.

[4]

4. Express $\cos 2x$ in terms of $\sin x$ only.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-T2 • Addition and double-angle formulae

1. WORKED ANSWER

Method: Use $\sin(45+30)$. Answer: $(\sqrt{6}+\sqrt{2})/4$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Use $\cos(45-30)$. Answer: $(\sqrt{6}-\sqrt{2})/4$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Use $\sin 2x = 2 \sin x \cos x$. Answer: $24/25$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Substitute $\cos^2 x = 1 - \sin^2 x$ into $\cos^2 x - \sin^2 x$. Answer: $1 - 2\sin^2 x$. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 15: Identities and wave form

Prove identities and convert $a \cos x + b \sin x$ to a single trigonometric function.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-T3. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Prove identities and convert $a \cos x + b \sin x$ to a single trigonometric function. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “Express $3 \cos x + 4 \sin x$ as $R \cos(x - \alpha)$, α acute.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 15: Build the method

Identities and wave form

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

Express $3\cos x + 4\sin x$ as $R\cos(x-\alpha)$, α acute.

WORKED REASONING

1. $R = \sqrt{3^2 + 4^2} = 5$.
2. Match $R\cos \alpha = 3$ and $R\sin \alpha = 4$.

Conclusion: $5\cos(x-53.1 \text{ degrees})$ approximately.

WORKED EXAMPLE 2

Find the maximum of $5\cos x + 12\sin x$.

WORKED REASONING

1. The amplitude after wave-form conversion is $\sqrt{25+144}$.

Conclusion: 13.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 15: Connect and reason

Identities and wave form

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-T3, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Prove $(1 - \cos 2x)/2 = \sin^2 x$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding identities and wave form.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-T3 • Identities and wave form

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. Express $3\cos x + 4\sin x$ as $R\cos(x-\alpha)$, α acute.

[4]

2. Find the maximum of $5\cos x + 12\sin x$.

[4]

3. Prove $(1-\cos 2x)/2 = \sin^2 x$.

[4]

4. Solve $3\cos x + 4\sin x = 5$ for $0 \leq x < 360$ degrees.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-T3 • Identities and wave form

1. WORKED ANSWER

Method: $R = \sqrt{3^2 + 4^2} = 5$. Match $R \cos \alpha = 3$ and $R \sin \alpha = 4$. Answer: $5 \cos(x - 53.1 \text{ degrees})$ approximately. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: The amplitude after wave-form conversion is $\sqrt{25 + 144}$. Answer: 13. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Both sides are defined for all real x . Answer: Use $\cos 2x = 1 - 2\sin^2 x$ and rearrange. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Convert to $5 \cos(x - 53.1) = 5$. Cosine equals 1 only when the angle is 0 modulo 360. Answer: x approximately 53.1 degrees. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 16: Integrated Higher geometry

Connect lines, circles, vectors and trigonometry in unfamiliar problems.

SPECIFICATION ALIGNMENT

Independent teaching summary for Qualifications Scotland Higher Mathematics: H-G7. This lesson develops conceptual understanding, communication using symbols, fluency, logical reasoning and strategic competence.

Understand the mathematical object

Connect lines, circles, vectors and trigonometry in unfamiliar problems. The central idea is that a representation is useful only when its meaning and conditions are explicit. In this higher geometric reasoning lesson, first identify the quantities, structures or events involved; then connect the words to notation, a diagram, a table or a graph. A reliable solution preserves that meaning at every step and ends with an answer that fits the original context.

KEY VOCABULARY

gradient; locus; tangent; vector; scalar product; radian; identity; transformation; proof; verification

Three controlling ideas

- Translate the diagram or statement into labelled mathematical relationships.
- Select a theorem or formula only after checking its conditions and units.
- Calculate exactly where possible, then verify direction, scale, geometry and accuracy.

PRIOR-KNOWLEDGE CHECK

Can you state the meaning of the key terms, identify the operation or relationship in “A circle has centre $(2, -1)$ and is tangent to $y = 2x + 7$. Find its radius.”, and predict the form or approximate size of the answer? Repair any weakness before continuing.

CHECKPOINT

Explain one method condition in a complete sentence. Give a simple valid example and one non-example that breaks the condition.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 16: Build the method

Integrated Higher geometry

Reliable routine

- Interpret: name the target, given information, conditions and units.
- Represent: write an equation, diagram, table, graph or exact numerical structure.
- Execute: show valid transformations or calculations in a readable order.
- Verify: test size, sign, units, domain, accuracy and contextual meaning.

WORKED EXAMPLE 1

A circle has centre $(2, -1)$ and is tangent to $y = 2x + 7$. Find its radius.

WORKED REASONING

1. Write the line as $2x - y + 7 = 0$.
2. Use point-to-line distance $|4 + 1 + 7|/\sqrt{5}$.

Conclusion: $12/\sqrt{5}$.

WORKED EXAMPLE 2

Find the point dividing $A(-1, 2, 4)$ to $B(8, 5, -2)$ in ratio 2:1.

WORKED REASONING

1. Use $(A + 2B)/3$.

Conclusion: $(5, 4, 0)$.

SELF-EXPLANATION PROMPT

Which line contains the main mathematical decision? Which line checks interpretation, reasonableness or accuracy? Explain how the method would change if one condition changed.

COMPREHENSIVE LESSON NOTEBOOK

Chapter 16: Connect and reason

Integrated Higher geometry

Why the method works

The method works because each stage preserves the defining relationship in the problem. A calculation is not merely a sequence of button presses: it is a chain in which every symbol, operation and representation has a stated meaning. An independent check uses a different feature - such as an inverse operation, an alternative representation, a bound, a unit or a limiting case - so it can reveal an error rather than repeat it.

COMMON MISCONCEPTION

A common error is to use a familiar rule before checking its conditions. For H-G7, this can produce a plausible-looking answer with the wrong structure, scale, sign, unit or interpretation. State the condition first, then apply the rule.

GUIDED REASONING TASK

Vectors a and b have magnitudes 5 and 7 with angle 60 degrees. Find $|a-b|$. Before calculating, write: (a) what is known, (b) what is required, (c) the controlling relationship, and (d) one check. Then complete the solution.

COMMUNICATION STANDARD

Use one mathematical statement per line. Define symbols, label units and diagrams, retain exact values until approximation is requested, and close with a sentence that answers the actual question.

EXTENSION OR REPAIR

Change one value or condition in the guided task. Predict which steps remain valid and which must change. This is the difference between memorising a routine and understanding integrated higher geometry.

COMPREHENSIVE LESSON NOTEBOOK

Independent practice

H-G7 • Integrated Higher geometry

Complete without notes. Show the setup, decisive working and final conclusion. Marks are independent teaching allocations, not official Qualifications Scotland / former SQA marks.

1. A circle has centre $(2, -1)$ and is tangent to $y = 2x + 7$. Find its radius.

[4]

2. Find the point dividing $A(-1, 2, 4)$ to $B(8, 5, -2)$ in ratio $2:1$.

[4]

3. Vectors $\mathbf{a}$ and $\mathbf{b}$ have magnitudes 5 and 7 with angle 60° . Find $|\mathbf{a} - \mathbf{b}|$.

[4]

4. A sector has area 54 cm^2 and arc length 12 cm. Find radius and angle in radians.

[4]

REFLECTION

Record the question number of your least secure answer. Name the error type and write the first corrected line that would repair it.

COMPREHENSIVE LESSON NOTEBOOK

Worked answers and repair notes

H-G7 • Integrated Higher geometry

1. WORKED ANSWER

Method: Write the line as $2x - y + 7 = 0$. Use point-to-line distance $|4 + 1 + 7|/\sqrt{5}$. Answer: $12/\sqrt{5}$. Check: confirm the result against the original conditions, units and expected scale.

2. WORKED ANSWER

Method: Use $(A + 2B)/3$. Answer: $(5, 4, 0)$. Check: confirm the result against the original conditions, units and expected scale.

3. WORKED ANSWER

Method: Use $|a - b|^2 = |a|^2 + |b|^2 - 2a \cdot b$ and $a \cdot b = 35\cos 60$. Answer: $\sqrt{39}$. Check: confirm the result against the original conditions, units and expected scale.

4. WORKED ANSWER

Method: Use $A = \frac{1}{2}rs$ to find r . Then $\theta = s/r$. Answer: $r = 9$ cm and $\theta = \frac{4}{3}$ radians. Check: confirm the result against the original conditions, units and expected scale.

COMPREHENSIVE LESSON NOTEBOOK

Geometry and Trigonometry mastery review

Explain, demonstrate, connect and revisit.

I can...

- H-G1 Parallel and perpendicular lines: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-G2 Gradient and inclination: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-G3 Medians, altitudes and perpendicular bisectors: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-G4 Equations of circles: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-G5 Tangents to circles: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-G6 Line-circle intersections: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-V1 Vector pathways: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-V2 Collinearity: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-V3 Internal division points: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-V4 Scalar product: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-V5 Angles between vectors: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-V6 Unit vectors and bases: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-T1 Trigonometric equations in degrees and radians: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-T2 Addition and double-angle formulae: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-T3 Identities and wave form: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.
- H-G7 Integrated Higher geometry: explain the idea, reproduce a valid method, solve an unfamiliar problem and check the result.

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